# Numerical Optimization

## Exercise IV

Jonathan Balzer

Geometric Modeling and Scientific Visualization Center
King Abdullah University of Science and Technology

# Exercises

1. Given $m$ tuples $(t_k, \boldsymbol{c}_k)$, $k = 1, \dots, m$, with $t_k \in \mathbb{R}$ and $\boldsymbol{c}_k \in \mathbb{R}^d$. Find the parametrized monomial curve

$$\boldsymbol{c}(t) = \sum_{i=0}^{n} \boldsymbol{p}_i t^i$$

of degree $n$ which best approximates the $\boldsymbol{c}(t_k) = \boldsymbol{c}_k$ in a least-squares sense.

# Exercises

2. Parametrize

$$\Phi := \{\boldsymbol{x} = (x, y, z)^\top \mid 1 - x^2 - y^2 - z^2 = 0, z \geq 0\}$$

by stereographic projection and find the point on $\Phi$ which is closest to $\tilde{x} = (0, 0, 2)^\top$ by the Gauss-Newton method.

# Exercises

3. Modify $\Phi$ to

$$\Phi_f = \{\boldsymbol{x} = (x, y, z)^\top \mid 1 - x^2 - y^2 - z^2 \geq 0, z \geq 0\}$$

and regard it as the feasible region of a constrained optimization problem. Calculate the

(i) active sets of $\boldsymbol{x}_0 = \mathbf{0}$, $\boldsymbol{x}_1 = (1/4, 0, 1/2)^\top$ and $\boldsymbol{x}_2 = (1, 0, 0)$,

(ii) tangent cone $T(\boldsymbol{x}_2)$,

(iii) and set of feasible directions $\mathcal{F}(\boldsymbol{x}_0)$.

# The Karush-Kuhn-Tucker (KKT) Condition

## Theorem

*Let $\boldsymbol{x}^*$ be a local solution of the canonical constrained minimization problem. If linear independence constraint qualification holds in $\boldsymbol{x}^*$, then there exists a Langrangian multiplier vector $\boldsymbol{\lambda}^*$ such that:*

(1a) $$\nabla_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) = \mathbf{0},$$

(1b) $$c_i(\boldsymbol{x}^*) = 0 \quad \forall i \in \mathcal{E}, \quad c_i(\boldsymbol{x}^*) \geq 0 \quad \forall i \in \mathcal{I},$$

(1c) $$\lambda_i^* \geq 0 \quad \forall i \in \mathcal{I},$$

(1d) $$\lambda_i^* c_i(\boldsymbol{x}^*) = 0.$$

# The Karush-Kuhn-Tucker (KKT) Condition

## Essence of proof.

1. $\nabla f(\boldsymbol{x}^*)^\top \boldsymbol{d} \geq 0$ for all $\boldsymbol{d} \in T(\boldsymbol{x}^*) = \mathcal{F}(\boldsymbol{x}^*)$.
2. Farkas: $\nabla f$ is in cone spanned by $\nabla c_i$, $i \in \mathcal{A}(\boldsymbol{x}^*) \Rightarrow$ (1a).
3. $\boldsymbol{x}^*$ feasible $\Rightarrow$ (1b).
4. Set $\lambda_i^* = 0$ for all $i \in \mathcal{I} \setminus \mathcal{A}(\boldsymbol{x}^*) \Rightarrow$ (1c), (1d).

□

## Definition

The condition $\lambda_i^* > 0$ for each $i \in \mathcal{A}(\boldsymbol{x}^*) \cap \mathcal{I}$ is called *strict complementarity*.