Numerical Optimization

Exercise V

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The Simplex Method

Linear program in standard form

Minimize $f : \mathbb{R}^n \mapsto \mathbb{R}$,

$$f(\boldsymbol{x}) = \boldsymbol{c}^\top \boldsymbol{x},$$

subject to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$, where

$$\mathbf{A} \in \mathbb{R}^{m \times n}, \quad \boldsymbol{b} \in \mathbb{R}^n, \quad \boldsymbol{c} \in \mathbb{R}^n, \quad m < n,$$

and $\mathbf{A}$ has full row rank.

Definition (Simplex)

The set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \boldsymbol{x} = \sum_{i=1}^{d} \lambda_i \boldsymbol{x}_i,\ \sum_{i=1}^{d} \lambda_i = 1,\ \boldsymbol{\lambda} \geq 0,\ \boldsymbol{x}_i \in \mathbb{R}^n \right\}$$

is called d-dimensional *simplex*, $d \in \{0, \ldots, n-1\}$, or simply *d-simplex*.

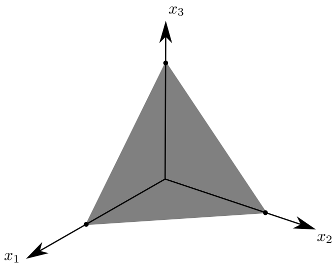
The Simplex Method

So why the name "simplex method"?

Rows of $\mathbf{A}$ define hyperplanes in $\mathbb{R}^n$. Their intersection restricted to the positive orthant, i.e. the feasible region, is an $n-m$-simplex (if non-empty and bounded).

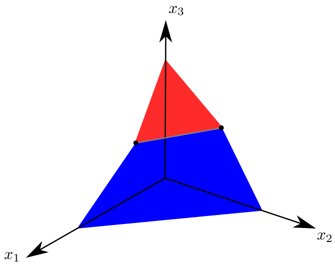
Example: $n = 3$

- $m = 1$: Picture the infeasible and unbounded case!

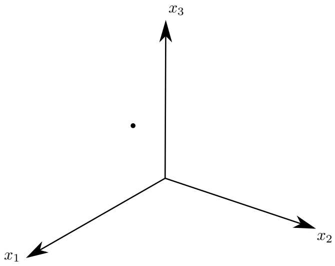


The Simplex Method

- $m = 2$: If both planes are not entirely infeasible, intersection non-empty due to rank assumption.



- $m = 3$: At most one feasible point if $\mathbf{A}$ has full row rank $\Rightarrow$ problem trivial.



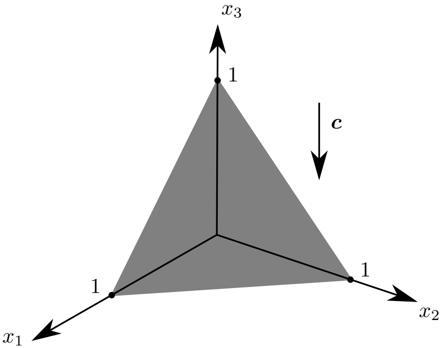
The Simplex Method

Standard simplex gives standard example:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix},$$

$$\boldsymbol{b} = 1,$$

$$\boldsymbol{c} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$



The Simplex Method

A step-by-step recipe:

1. Start on face of positive orthant, say in the x_1x_3-plane:

$$\mathcal{B} = \{1\}, \quad \mathcal{N} = \{2, 3\},$$

$$\mathbf{B} = 1, \quad \mathbf{N} = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

$$\boldsymbol{x}_B = x_1 = \mathbf{B}^{-1}\boldsymbol{b} = 1, \quad \boldsymbol{x}_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The Simplex Method

2. Dual variables $\boldsymbol{\lambda}, \boldsymbol{s}$ from

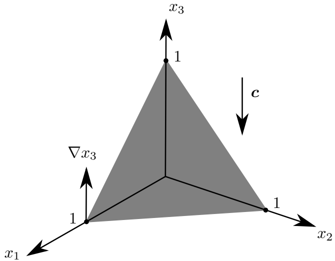
   - complementary slackness $\boldsymbol{s}^\top \boldsymbol{x} = 0$

$$\boldsymbol{s}_B = s_1 = 0$$

   - Lagrange condition $\mathbf{A}^\top \boldsymbol{\lambda} + \boldsymbol{s} = \boldsymbol{c}$:

$$\boldsymbol{\lambda} = \mathbf{B}^{-\top} \boldsymbol{c}_B = c_1 = 0,$$

$$\boldsymbol{s}_N = \boldsymbol{c}_N - \mathbf{N}^\top \boldsymbol{\lambda} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$



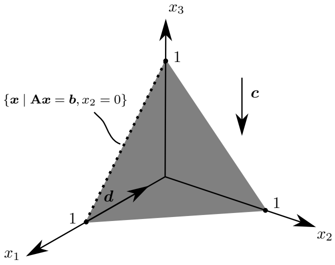
The Simplex Method

3. If $\boldsymbol{s} \geq \mathbf{0}$, terminate, else select entering index, here $q = 3$.

4. Deactivate $x_q \geq 0$: Increase x_q in $\boldsymbol{x}_N$ by "decreasing" $\boldsymbol{x}_B$ in the direction

$$\boldsymbol{d} = \mathbf{B}^{-1}\mathbf{A}_q = 1$$

(coupling due to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$).



The Simplex Method

5. Let $j \in \{1, \ldots, |\mathcal{B}|\}$ be the row index of $\boldsymbol{x}_B$, for which 0 is reached first, i.e. $x_{B,j} - \alpha d_j = 0$ for some $\alpha \in \mathbb{R}$. Turns out that

$$\alpha = \min_{i \in \{1,\ldots,|\mathcal{B}|\}} \frac{x_{B,i}}{d_i} = 1 = x_q$$

so that

$$\boldsymbol{x}_B^+ = 1 - 1 = 0, \quad \boldsymbol{x}_N^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

6. Map j to leaving index p, update

$$\mathcal{B} = \{p\} = \{3\}, \quad \mathcal{N} = \{q, 2\} = \{1, 2\},$$
$$\mathbf{B} = 1, \quad \mathbf{N} = \begin{pmatrix} 1 & 1 \end{pmatrix},$$
$$\boldsymbol{x}_B = x_q = x_3 = 1, \quad \boldsymbol{x}_N = \begin{pmatrix} x_2 \\ x_p \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(switch x_p in $\boldsymbol{x}_B^+$ and x_q in $\boldsymbol{x}_N^+$), and continue with 2.

Exercises

1. Consider the linear program min $-5x_1 - x_2$ subject to

$$\begin{aligned} x_1 + x_2 &\leq 5, \\ 2x_1 + \frac{1}{2}x_2 &\leq 8, \\ \boldsymbol{x} &\geq \mathbf{0}. \end{aligned}$$

(i) Bring it to standard form.

(ii) Use the simplex method to solve it.