Numerical Optimization

Exercise VII

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Quadratic Programming

Equality-constrained case

Minimize $q : \mathbb{R}^n \mapsto \mathbb{R}$,

$$q(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^\top \mathbf{G}\boldsymbol{x} + \boldsymbol{c}^\top \boldsymbol{x},$$

subject to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{G} \in \mathbb{R}^{n\times n}, \quad \boldsymbol{c} \in \mathbb{R}^n, \quad \mathbf{A} \in \mathbb{R}^{m\times n}, \quad \boldsymbol{b} \in \mathbb{R}^n, \quad m < n,$$

and $\mathbf{A}$ has full row rank.

KKT system:

$$\begin{pmatrix} \mathbf{G} & -\mathbf{A}^\top \\ \mathbf{A} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{c} \\ \boldsymbol{b} \end{pmatrix}$$

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Null space method:

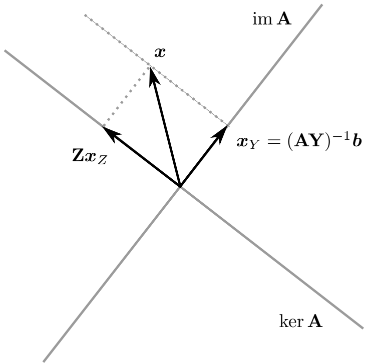
- distinguish image and null space coordinates of solution:

$$\boldsymbol{x} = \mathbf{Y}\boldsymbol{x}_Y + \mathbf{Z}\boldsymbol{x}_Z$$

- feasible base point:

$$\boldsymbol{x}_Y = (\mathbf{AY})^{-1}\boldsymbol{b}$$

(from linear constraint)



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- Lagrange condition:

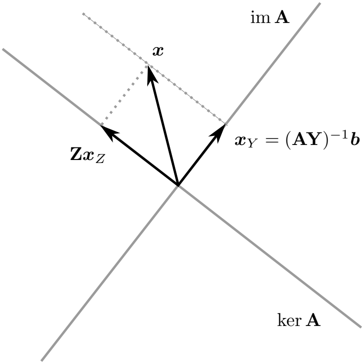
$$\mathbf{G}\mathbf{Z}\boldsymbol{x}_Z = \mathbf{A}^\top \boldsymbol{\lambda} - \mathbf{G}\mathbf{Y}\boldsymbol{x}_Y - \boldsymbol{c}$$

- after multiplication with $\mathbf{Z}^\top$:

$$\mathbf{Z}^\top \mathbf{G}\mathbf{Z}\boldsymbol{x}_Z = -(\mathbf{Z}^\top \mathbf{G}\mathbf{Y}\boldsymbol{x}_Y + \mathbf{Z}^\top \boldsymbol{c})$$

since

$$\mathbf{Z}^\top \mathbf{A}^\top = (\mathbf{A}\mathbf{Z})^\top = \mathbf{0}$$



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Other interpretation:

Insert $\boldsymbol{x} = \mathbf{Y}\boldsymbol{x}_Y + \mathbf{Z}\boldsymbol{x}_Z$ into original to obtain *reduced* problem

$$\min_{\boldsymbol{x}_Z} \frac{1}{2}\boldsymbol{x}_Z^\top \tilde{\mathbf{G}} \boldsymbol{x}_Z + \tilde{\boldsymbol{c}}^\top \boldsymbol{x}_Z$$

with

$$\tilde{\mathbf{G}} = \mathbf{Z}^\top \mathbf{G} \mathbf{Z} \quad \text{and} \quad \tilde{\boldsymbol{c}} = \mathbf{Z}^\top \mathbf{G} \mathbf{Y} \boldsymbol{x}_Y + \mathbf{Z}^\top \boldsymbol{c}.$$

Consequences:

- equality-constrained problems can be converted into unconstrained ones
- inequality-constrained problems can be converted into equality-constrained ones ($\hookrightarrow$ active set method)

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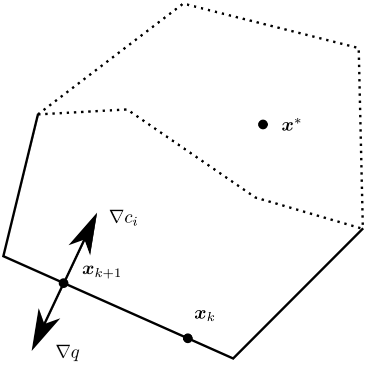
Active set method:

1. start with feasible point
2. solve problem in subset spanned by relevant constraints (*working set*)
3. if no blocking encountered, check KKT condition: terminate or deactivate constraint and go to 2.



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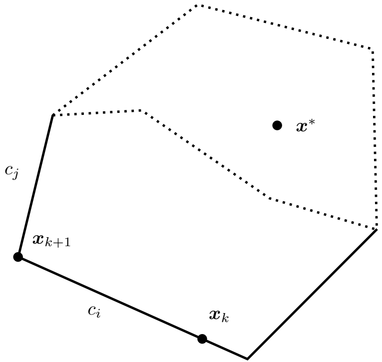
4. else add blocking constraint to working set

5. continue with 2.



Exercises

1. Consider the linear mapping $f : \mathbb{R}^3 \mapsto \mathbb{R}^2$ given by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Find a basis of the null space $\ker \mathbf{A}$.

Exercises

2. Minimize $q : \mathbb{R}^2 \mapsto \mathbb{R}$,

$$q(x_1, x_2) = x_1^2 + 2x_2^2 + 2x_1x_2,$$

subject to the constraint

$$x_1 + 3x_2 = 2$$

by the null space method. Compare the result to the one obtained using the constraint to eliminate x_1 from $q(x_1, x_2)$.